2011 DOE Vehicle Technologies KIVA-Development

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Los Alamos National Laboratory

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Project ID # ACE014

LA-UR-11-10120

Overview

Timeline

- 10/01/09
- 09/01/13
- 40% complete

Budget

- Total project funding to date:
  - 1,140K
  - Contractor share 30%
- Funding received in FY10
  - 580K
  - Funding for FY11 – 660K
    - Received only 1/3
    - Continuing Resolution restricting ability to meet milestones and progress

Barriers

- Improve understanding of the fundamentals of fuel injection, fuel-air mixing, thermodynamic combustion losses, and in-cylinder combustion/ emission formation processes over a range of combustion temperature for regimes of interest by adequate capability to accurately simulate these processes
- Engine efficiency improvement and engine-out emissions reduction
- Minimization of engine technology development

Partners

- Iowa State University - Dr. Song-Charng Kong (under funded)
- University of Nevada, Las Vegas - Dr. Darrell W. Pepper (not yet contracted FY 11)
- University of Purdue, Calumet - Dr. Xiuling Wang (under funded)
- University of New Mexico- Dr. Juan Heinrich (FY 11 - not yet contracted)

FY 09 to FY 13 KIVA-Development

Objectives

- **Robust, Accurate Algorithms in a Modular setting** –
  - **Relevance to accurately predicting engine processes to enable better understanding of, flow, thermodynamics, sprays, etc….**
    - Developing more robust and accurate algorithms for helping to understand better combustion processes in internal engines
    - Providing a better mainstay tool for improving engine efficiencies and help in reducing undesirable combustion products.
    - Newer and mathematically rigorous algorithms will allow KIVA to meet the needs of future and current combustion modelers and designers.
    - Developing Fractional Step (PCS) Petrov-Galerkin (P-G) and Characteristic-Based Split (CBS) *hp*-adaptive finite element method
    - Conjugate Heat Transfer for providing more accurate prediction in wall-film and its effects on combustion and emissions under PCCI conditions with strong wall impingement. Providing accurate boundary conditions.
- **Easier and quicker grid generation**
  - **Relevant to minimizing of engine technology development**
    - Cut-Cell grid implementation: CAD to CFD
    - Cut-cell output to KIVA-4 via Cubit and Cubit to KIVA converter.

Milestones for FY 10- FY11

06/09- Started Researching Fractional Step CBS method – developed some research coding and altering/combining with existing Projection code and *h*-adaptive algorithms.

09/09 – 2D and 3D P-G Fractional Step (PCS/CBS) Finite Element Algorithm Developed (mathematics, engineering documents and evaluation).

01/10 – *h*-adaptive grid technique/algorithm implement in PCS/CBS-FEM method for 2D

02/10 – *h*-adaptive grid technique/algorithm implement in PCS/CBS-FEM method for 3D

02/10 – *hp*-adaptive FEM Algorithm & Framework: continued development and changes.

02/10 thru 09/10 –

Successful at meeting standard benchmark problems in the incompressible regime using CBS Fractional Step (PCS) and P-G stabilization without CBS stabilization.

04/10 – New algorithm for **Cut-Cell** grid generation started – more robust algorithm.

05/10 – Multi-Species Transport testing in PCS/CBS-FEM algorithm.

09/10 – 2D and 3D Characteristic-based Split (CBS) stabilization extended to turbulence closure routines. Continue debugging CBS and combination of CBS and PG capabilities.

10/10 – P-G found to be more flexible than CBS stabilization via benchmark comparisons.

12/10 – Start looking into Runga-Kutta method for 2nd order-in-time in P-G only Fractional Step.

12/10 – Benchmark tests of PCS and CBS and begin comparing results.

Find CBS a bit less flexible the P-G but, does provide good solution, and can be used with P-G.

12/10 – Inserting PCS/CBS algorithm/coding **into *hp*-adaptive Framework**.

01/11 – FY11 Engineering documentation and precise algorithm details.

02/11 – Continue working the **PCS/CBS** method in the **compressible flow** regimes.

02/11 – Discussions with Sandia on Cut-cell (our algorithm) & incorporation into **Cubit** software.

Approach

- Approach for Developing Robust and Accurate Numerical Simulation Code:
  - Computational Physics
    - Understanding of the physical processes to be modeled
    - Assumptions inherent in any particular model
      - Ability of the chosen method, the mathematical formulation, and its discretization to model the physical system to within a desired accuracy.
    - The ability of the models to meet and or adjust to users' requirements – modularity, documentation.
    - The ability of the discretization to meet and or adjust to the changing needs of the users.
    - Validation and Verification (V&V) – meeting requirements and data.
    - Effective modeling employs good software engineering practices.

Development Approach and Milestones

- Approach for Robust and Accurate Numerical Simulation:
  - Development Process
    - Understanding of the physical processes to be modeled
      - Mathematical representations and evaluation of appropriate methods and models.
      - Guiding engineering documents
        - Assumptions inherent in particular model and methods
        - Ability of hp-adaptive PCS/CBS method, the mathematical formulation, and its discretization to model the physical system to within a desired accuracy.
        - The ability of the models to meet and or adjust to users' requirements – chose
        - The ability of the discretization to meet and or adjust to the changing needs of the users.
        - Effective modeling employs good software engineering practices.
          - Modularity, Documentation, Levelized (under-the-hood)
    - Validation and Verification (V&V) – meeting requirements and data.
      - Verification via known algorithm substitution
      - Validation and development process
        - Benchmark Problems that exercise all code in all flow regimes

FY-11 Technical Accomplishments

- Developing *hp*-adaptive PCS/CBS FEM Discretization for:
  Accurate and Robust Turbulence Reactive Flow Modeling – Combustion Modeling
- 2-D and 3-D PCS/CBS *h*-adaptive FEM codes are coded:
  - Benefit of Eulerian system with 2nd order-in-time algorithm
  - Performed without large system of linear equations to solve!
  - CBS or Petrov-Galerkin Stabilization (P-G) having 3rd order spatial accuracy
  - Numerical dispersion precisely measured and removed prior to solution advancement.
  - Various choices of Stabilizing Modes:
    - P-G with CBS for 2nd order-in-time or use CBS alone for 2nd order-in-time
  - 1 pressure solve per time step : Semi-implicit or an Explicit modality.
  - Equal-order: same basis for pressure and momentum (if desired).
  - *h*-adaptive with Residual error & Gradient control (incorporated FY 09).
  - *k-ω* turbulence model FY-09 & FY-10
  - *k-ε* blended low Reynolds (Wang, Carrington, Pepper 2009) .
  - New wall function system for both 2D and 3D - compressible (variable density in FY11).
  - PCG Solver & in-situ stationary preconditioning (FY 10)
  - New MAKE system (FY10).
  - Stochastic particle model - now porting KIVA-4 model to FEM method FY11/12.
  - Verification complete
    - Via known algorithm substitution and benchmark problems solution
  - Validation and continued development and error/bug removal via
    - Benchmarks Problems

FY-11 Technical Accomplishments

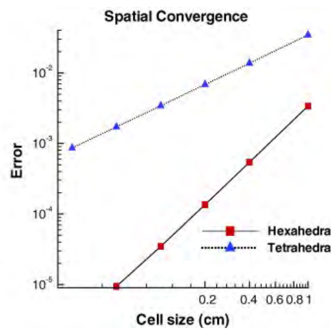
- Conjugate Heat Transfer (CHT)
  - Motivation
    - Extend KIVA-4 capability to predict heat conduction in solids.
    - Use KIVA-4 to perform simultaneous simulation of in-cylinder processes and heat conduction in mechanical components.
  - Expected outcome
    - Prediction of combustion chamber wall temperature distribution.
    - More accurate prediction of wall film and its effects on combustion and emissions under PCCI conditions with strong wall impingement.
  - Approach
    - Modify KIVA-4 for heat conduction calculation in solid.
    - Extend the computational domain to include both fluid and solid domains.
    - Perform integrated thermo-fluids modeling in one simulation using the same code.
    - Applicable energy equation is solved for temperature distribution in solids

Why Do We Need New Method? Accuracy & Robustness
KIVA-4 Spatial Convergence vs. FEM on Regular Grids

2011 DOE
Merit Review

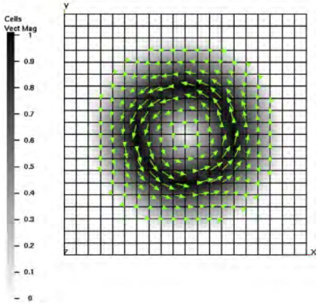
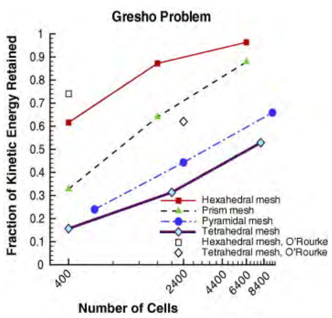
KIVA-4*

1-D diffusion

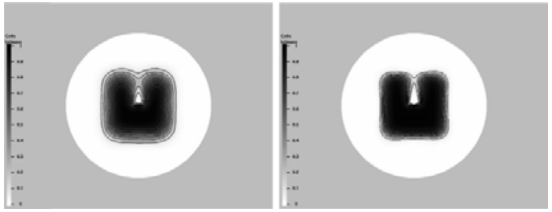


Slope for tets ~ 1

Gresho momentum flux

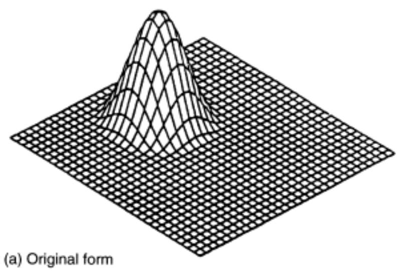
Rotating notch



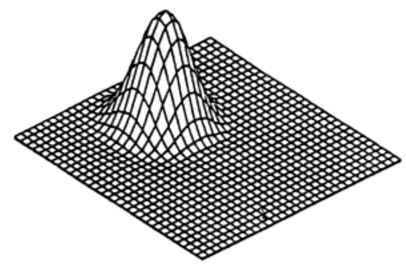
6,000 prisms   88,000 prisms

FEM- CBS**

Rotating Cosine hill similar to rotating notch



(a) Original form



(b) Form after one revolution using consistent M matrix

~225 cells

*used with permission from Dave Torres.

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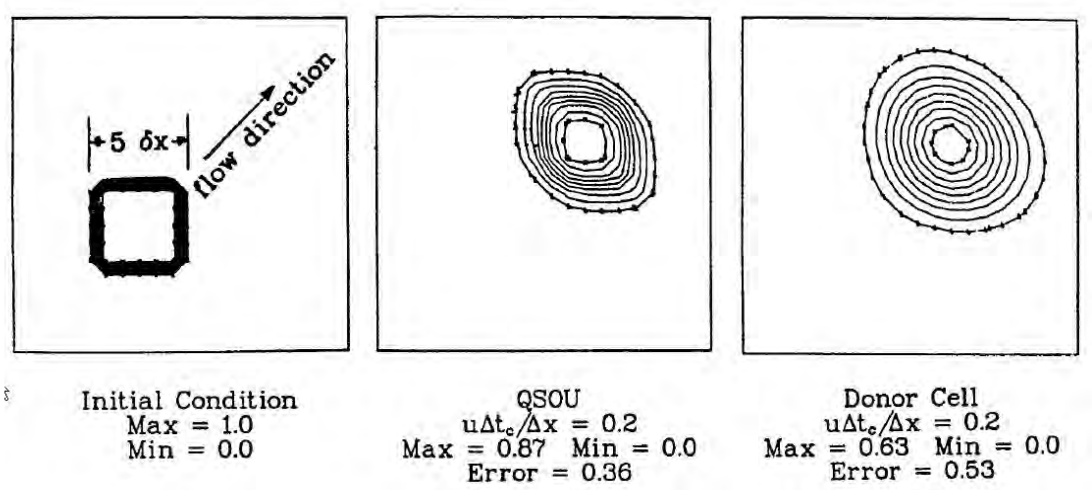
Why Do We Need New Method? Accuracy & Robustness
Eulerian advection: Current KIVA vs. FEM on Regular Grids

Current KIVA advective Flux*



Phase C advective flux is very diffusive

*from – KIVA-II manual: A Computer Program for Chemically Reacting Flows with Sprays, LA-11560-MS, Los Alamos Scientific Report, 1989.

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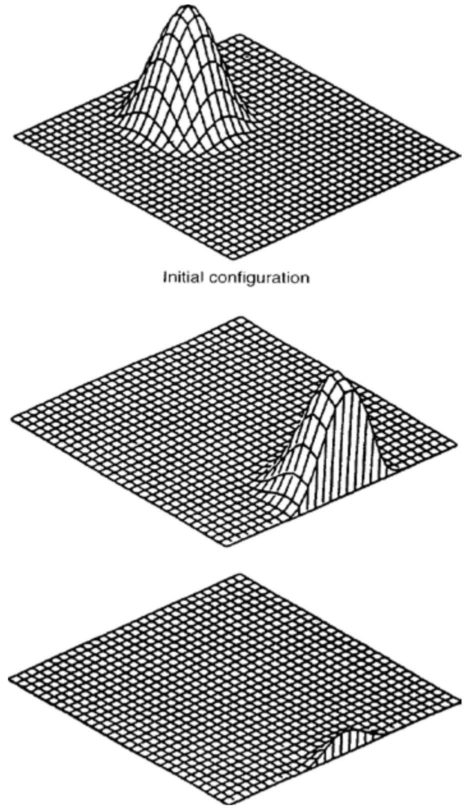
CBS-FEM Advection**

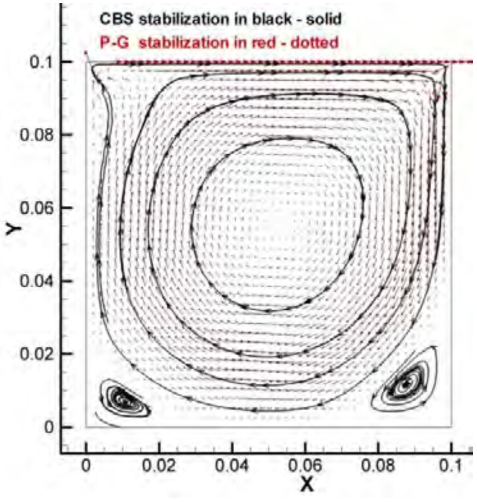


Advection is nearly exact!

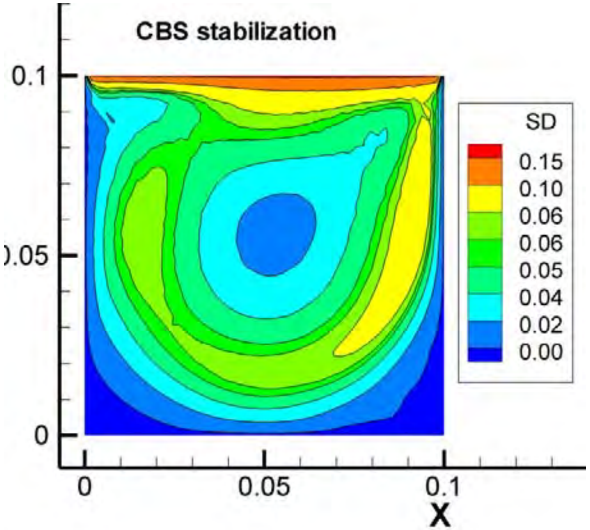
Added benefit of Eulerian Frame along with 2nd order-in-time without large linear equations to solve.

Validation of 2-D Fractional Step – FEM

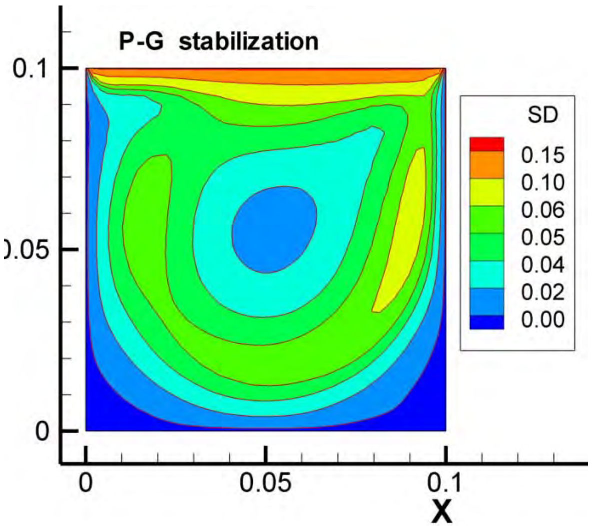
- Driven Cavity Benchmark – Re = 1000
  - Semi-implicit solve – pressure Poisson equation
  - **KIVA-4 published solution shows ~45,000 cells** for low Mach equations, an order magnitude larger than PCS or CBS FEM!
  - Characteristic vs. P-G :
    - P-G is more flexible, has good adjustment of element size h_e
    - Characteristic has 2nd O in time inherent in scheme
    - Adaptation at Pressure singularity in upper corners really helps solution



Grid 40x50



Characteristic stabilization

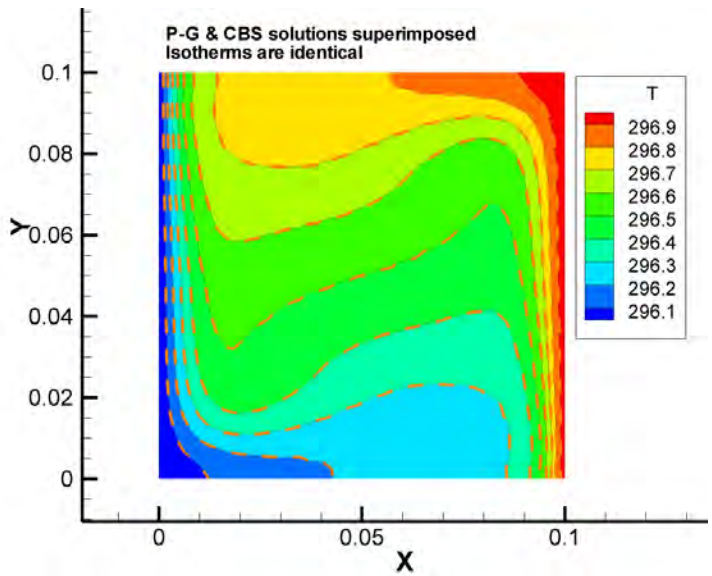


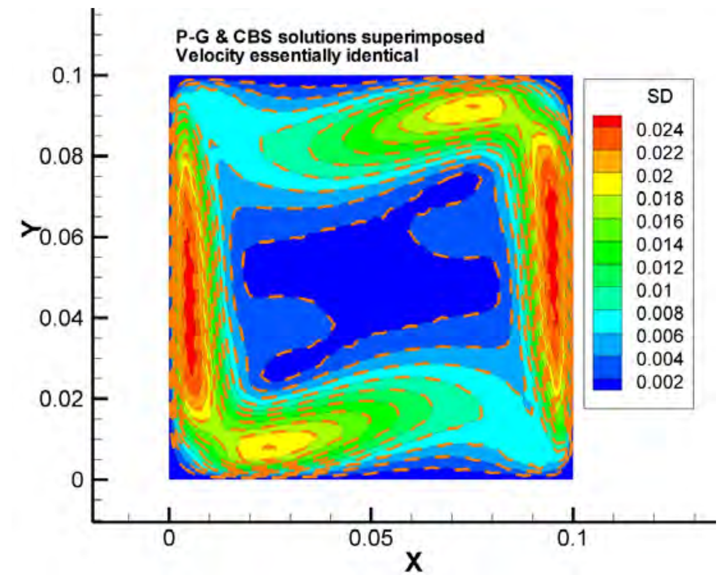
P-G stabilization

Validation of 2-D Characteristic-Based Split – FEM

- Slightly compressible low speed flow.
- Differentially Heated Cavity - Ra = 1.0e06.
- Pressure Poisson matrix solved.
- Identical results between CBS and P-G stabilization
  - Source Term (*Boussinesq approximation*) helps 1st order –in-time scheme be as accurate as 2nd order CBS method.

2 solutions: P-G and CBS stabilization

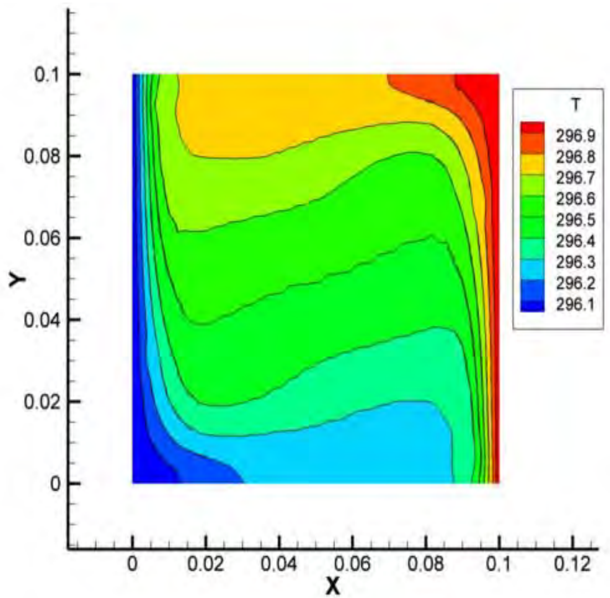




Isotherms 40x50 Grid Isotachs

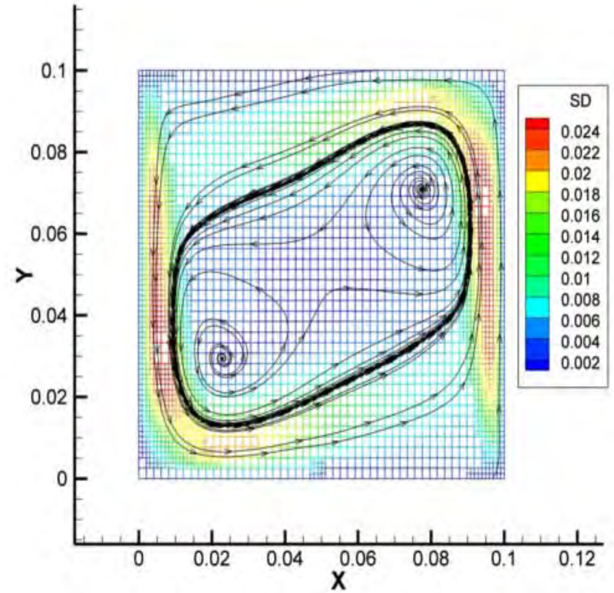
Validation of 2-D *h*-adaptive – PG PCS FEM

- Slightly compressible low speed flow.
- Differentially Heated Cavity - Ra = 1.0e06.
- Pressure explicit mode.
- P-G stabilization



Isotherms

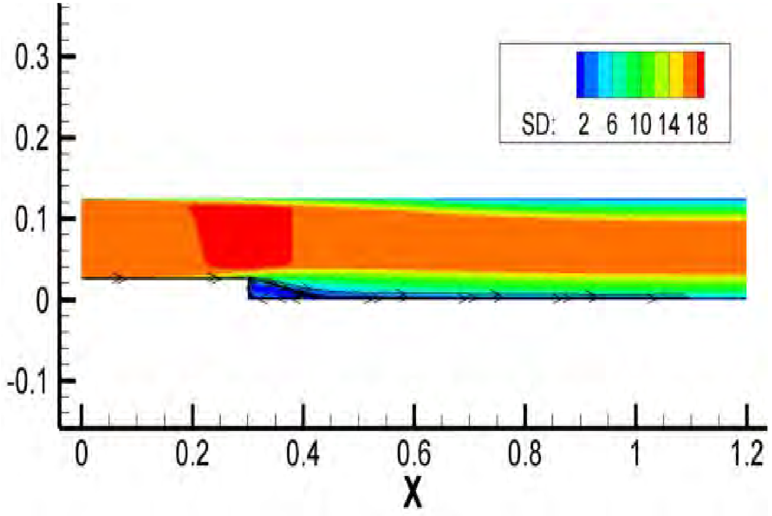
Adapted 40x50 Grid

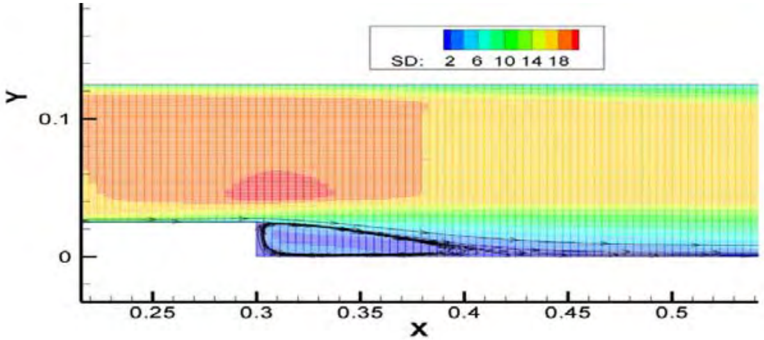


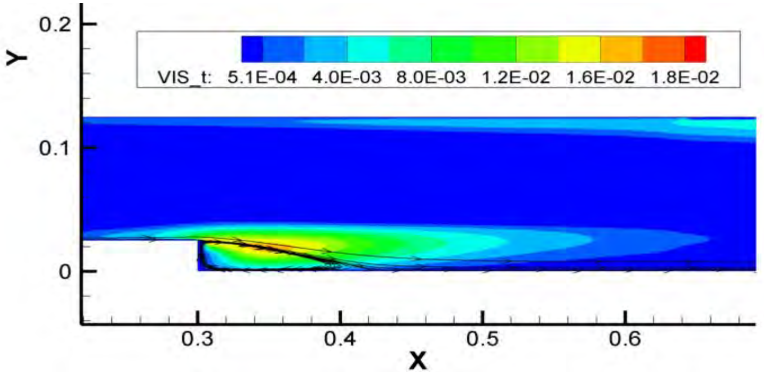
Isotachs

CBS/PG *h*-adaptive Validation

- Turbulent convective flow over a backward-facing step
  - (current KIVA can't do this problem well).
- Re=28,000, inflow is 17m/s (Mach number ~0.05) matches data.
  - Lower velocity in a typical internal combustion engine.
- 2 species at inlet with different mass fractions, both are air
  - multi-species testing.
- 1 specie at t=0.
  - Combined CBS and P-G stabilization
  - k-ω closure model
  - currently recirculation ~6.0h







Subsonic flow regime

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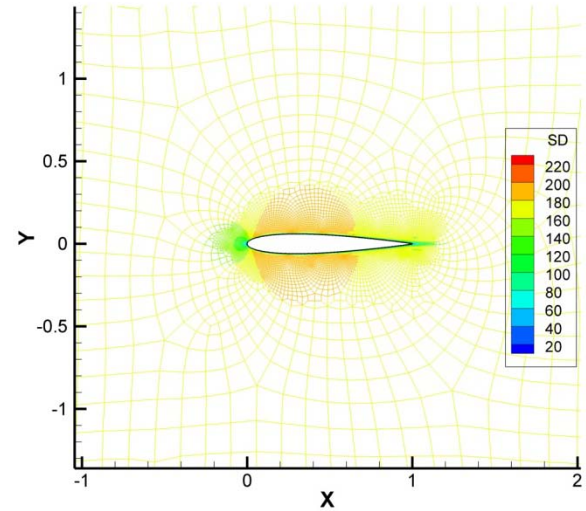
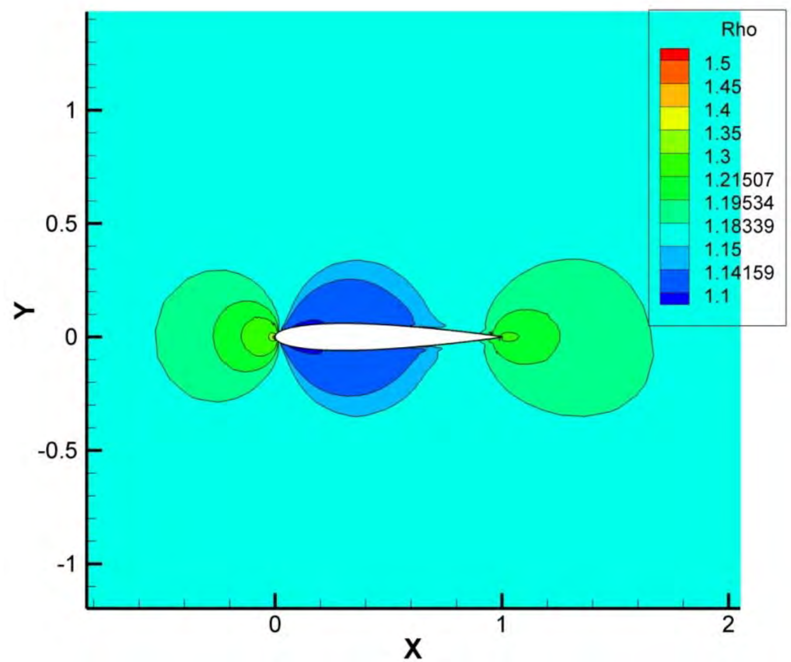
Merit Review

- Time dependent solution
  - CBS and P-G combined system.
  - Multi-species testing, 2 species at inlet.

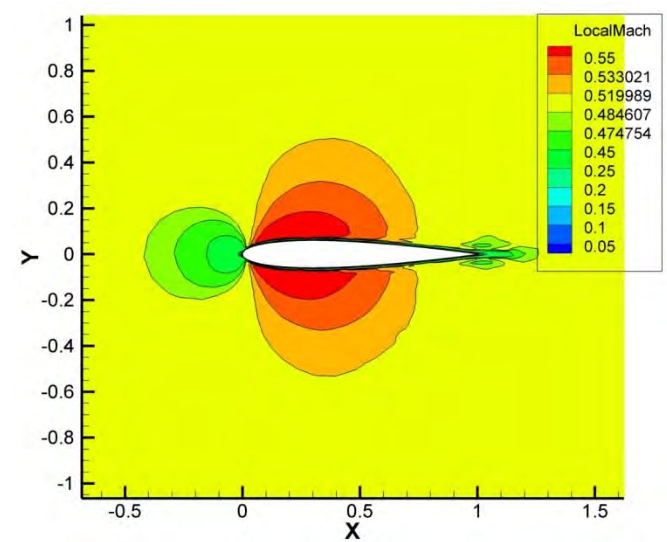


~8000 cells and nodes – adapted on boundary

NACA 0012 airfoil test
Mach = 0.5 & α = 0



Density



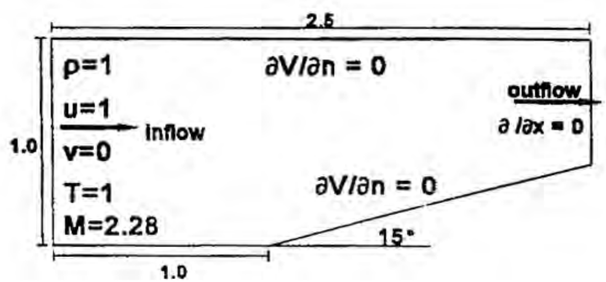
Local Mach Number

Density Contour on a Planar 15° Ramp at M=2.28

- 15° compression ramp
- *h*-adaptive P-G FEM
  - Explicit scheme with Runga-Kutta 2
- Time dependent solution
  - Our research code

    identical to CBS system in explicit mode.

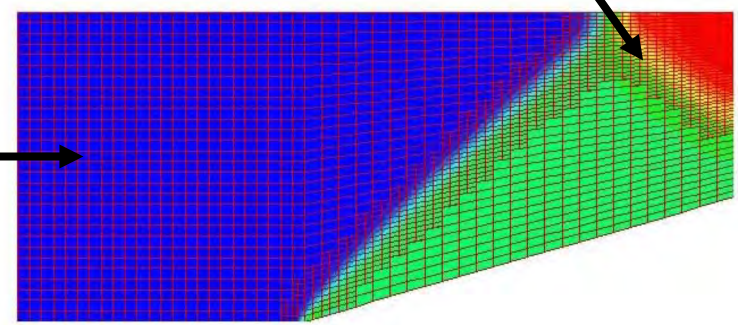
Mach = 2.28 at inlet



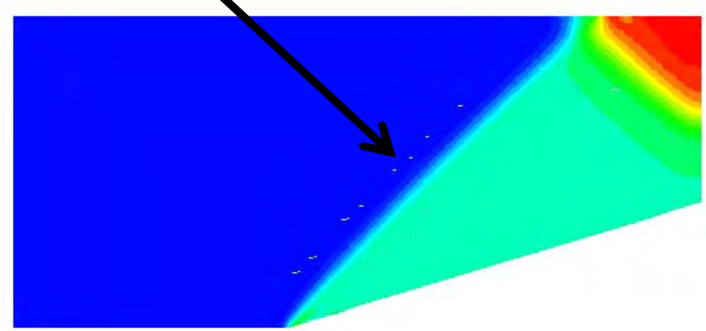
Boundary
Conditions

Reflection

Flow



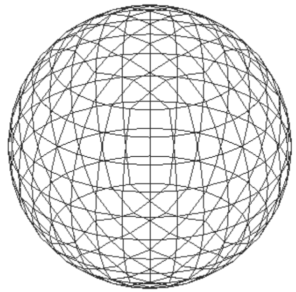
Shock



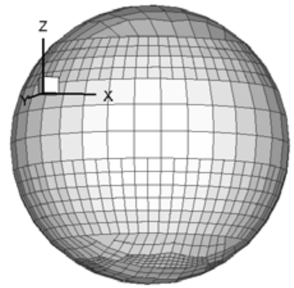
Natural Convection in a Differentially Heated Sphere

*hp-a*daptive FEM*

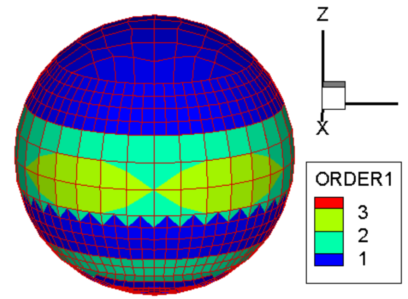
Ra=10[4]



(a) initial mesh



(b) intermediate *h*-adaptive mesh

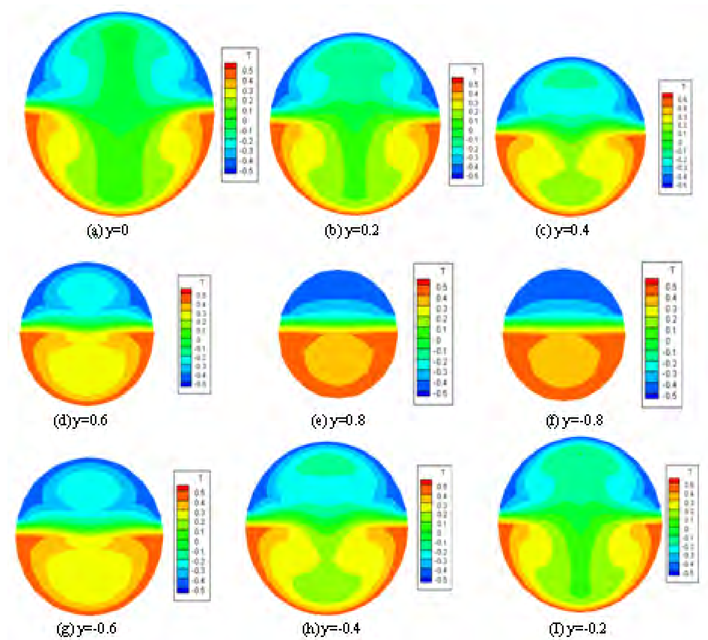


(c) final *hp*-adaptive mesh

- Demonstrating Solver Capability
  - Truly curved and complex domains

2-D planar isotherms at $-1 \leq R \leq 1$
-0.8, -0.6, -0.4, -0.2, 0.0, 0.2, 0.4, 0.6, 0.8
along major axis in x-z planes

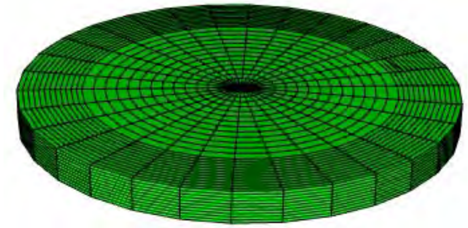
*used with permission from Wang, X. and Pepper, D. W.



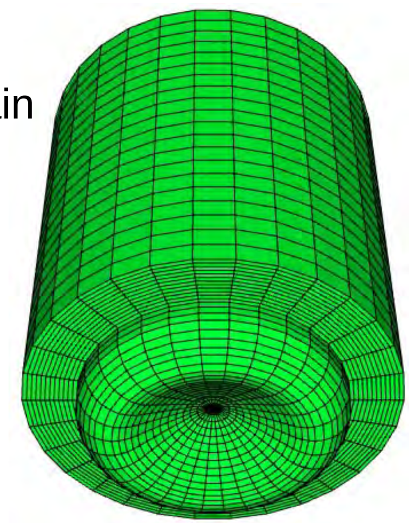
CHT Modeling – Current Progress

- Caterpillar engine cases
- Initial conditions for interfaces
  - T_piston=550 K
  - T_head=523 K
- Boundary conditions
  - T_piston=500 K
  - T_head=400 K
  - T_wall=433 K
- Spray conditions
  - t_inj=-7.0 CA, inj_dur=19.75 CAD

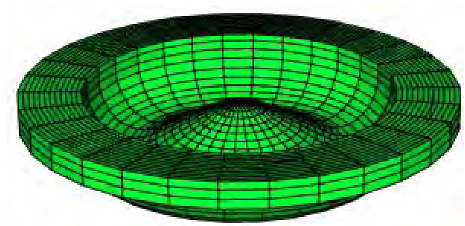
Head



Gas domain

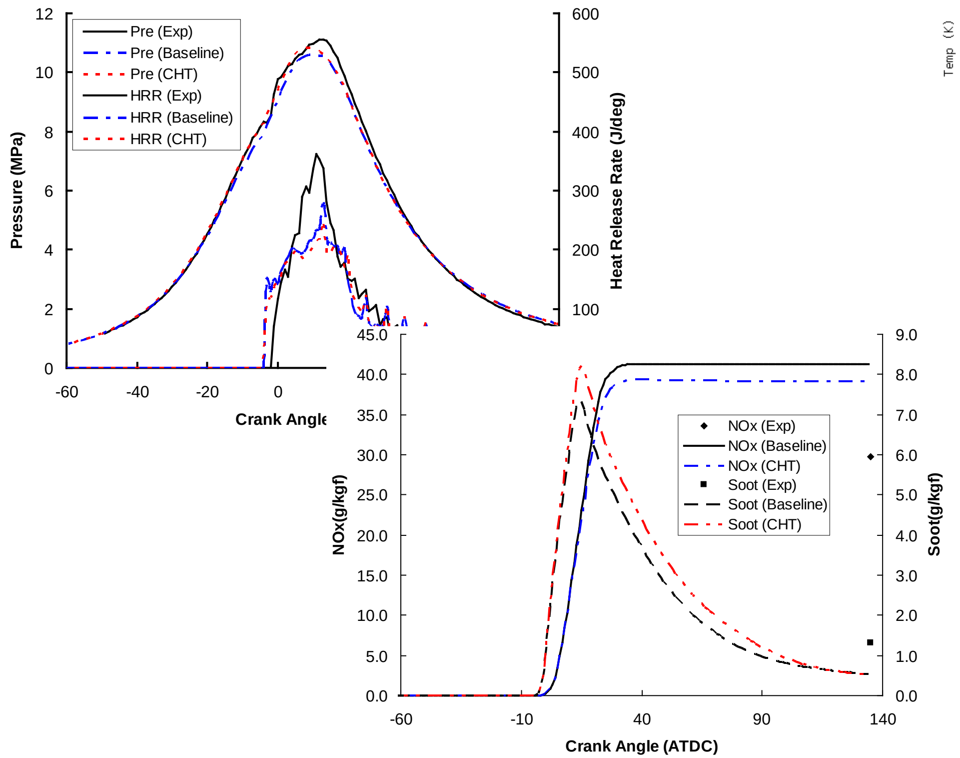


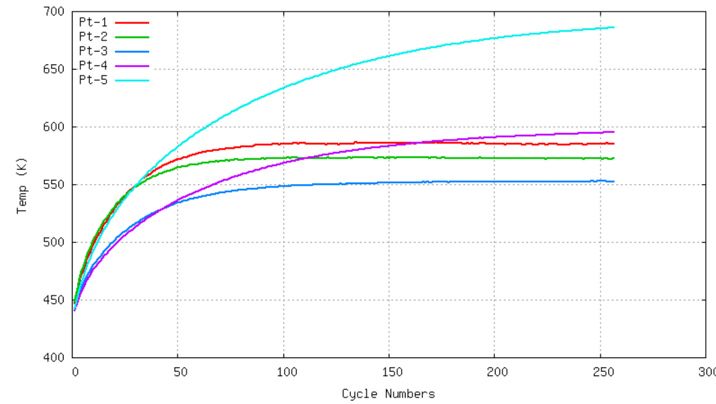
Piston

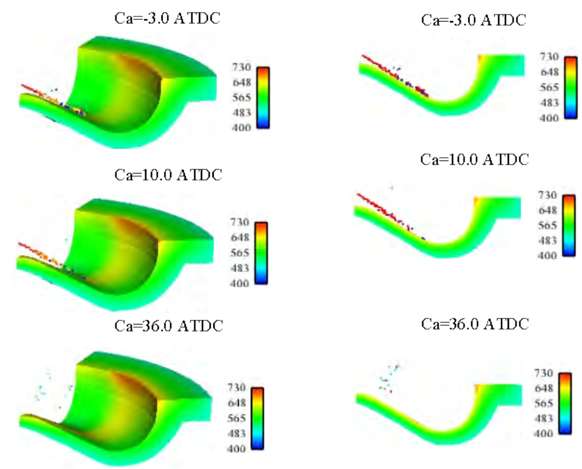


CHT - Cat Engine Test Results

- Overall combustion and emissions predictions are similar to the baseline case using uniform surface temperatures.
  - In general users are good at specify temperature and making adjustments in the models to produce good results on known systems.
- Nonetheless, CHT is able to predict the surface T distribution (thermal loading) in the combustion chamber.
  - More predictive modeling capability.





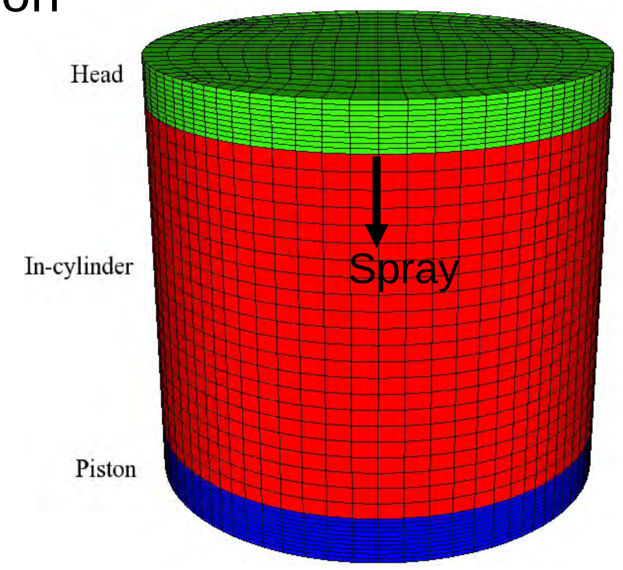


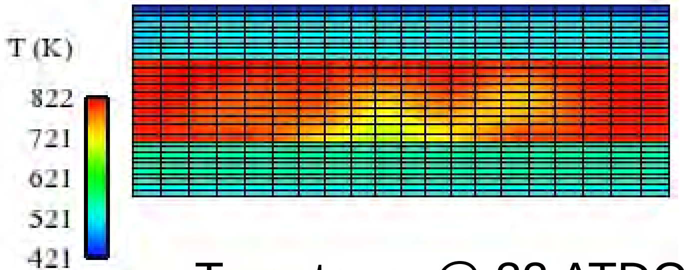
Enhancements to KIVA-4 MPI for CHT modeling in solids

•The code can run for both conventional mesh and CHT mesh

•CHT model validated via known analytic solution

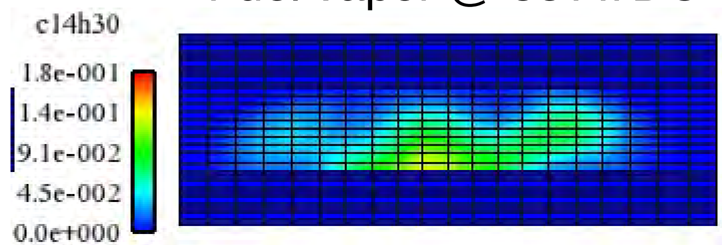
- Flat piston and cylinder head
- Initial conditions for interfaces
  - T_piston=550 K; T_head=523 K
- Boundary conditions
  - T_piston=500 K; T_head=400 K
- Spray conditions
  - t_inj=-9.0 CA, inj_dur=19.75 CAD
  - m_f=28 mg





T contours @ 33 ATDC

Fuel vapor @ 33 ATDC



Program Collaborators

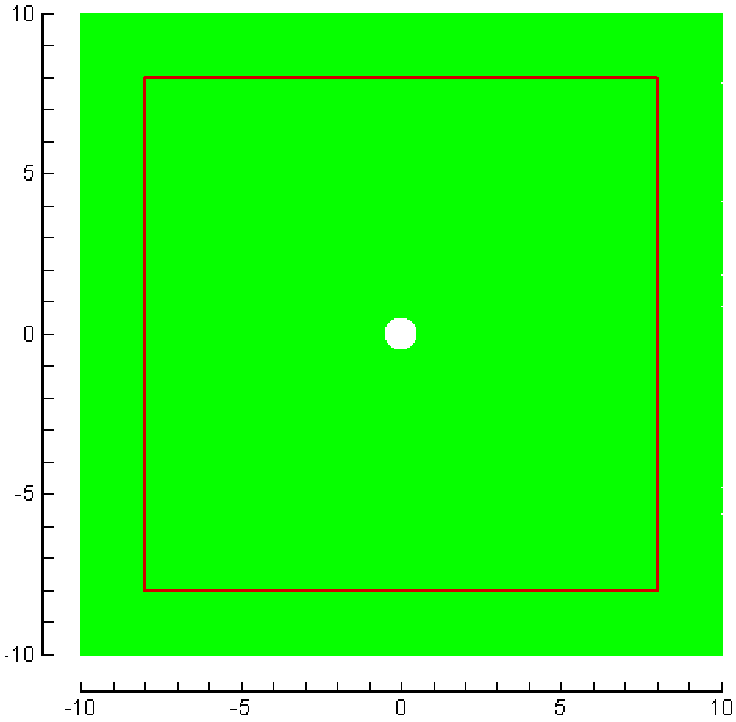
- LLNL collaborating
  - providing great feedback and reporting on KIVA-4mpi
- Iowa State University
  - Conjugate Heat Transfer in KIVA-4 and KIVA-4mpi
    - Song-Charng Kong & GRA and Postdoc.
- Purdue, Calumet
  - *hp*-Adaptive FEM with Characteristic-Based Split (CBS)
    - Xiuling Wang (Purdue) and GRA
- University of Nevada, Las Vegas
  - *hp*-Adaptive FEM with Characteristic-Based Split (CBS)
    - Darrell Pepper (UNLV) and GRA.
- University of New Mexico
  - Moving Immersed Body and Boundaries Algorithm Development
    - Juan Heinrich and Graduate Student

Future or Ongoing effort in FY11 to FY 13

- PCS/CBS-FEM
  - Test cases: finish tests (LANL & Purdue)
    - Simple unit, various benchmark problems and more complex domains too/
    - Make rigorous comparisons to data and analytics.
      - Publish results in peer reviewed articles.
  - Develop KIVA type I/O and interfacing.
  - Incorporate the injection/spray model and reactive chemistry coding.
  - Overset Grid method for moving parts. Moving grid – new algorithm development for moving boundaries and immersed bodies. Immersed moving bodies - UNM.
  - Mixed element types - UNLV.
  - Turbulence modeling – LANL, Purdue, UNLV.
  - Parallel constructions – Matrix solver already developed for massively parallel constructions (All).
- Conjugate Heat Transfer (CHT) modeling
  - Develop partitioning algorithms for solid domain for parallel computing
  - Perform simulation using multiple processors
  - Conduct combustion modeling
  - Test the code in practical bowl-in-piston geometry – challenges in partitioning complex geometry of the solid domain

Future – New Immersed and Moving Boundary Algorithm

- **Improving the current algorithms**
  - Increase robustness - generic method.
  - Simulations with higher resolution.
  - Use of overset parts/grids.
  - Good candidate:Unstructured grid, precisely locate body.
    - 2^{nd} Order in space.
    - Grid is of body only, fluid only.
    - Boundary condition update
  - Movie of ball/fluid interaction*
    - Juan Heinrich (University of New Mexico).



*Used with permission from Juan Heinrich

Summary

- **Accurate, Robust and well Documented algorithms**
  - Developing and implementing robust and extremely accurate algorithms in KIVA-4 architecture – PCS/CBS *hp-adaptive* FEM.
  - Reducing model's physical and numerical assumptions.
  - Measure of solution error: resolution when and where required.
  - New algorithm requiring less communication, no pressure iteration, an option for explicit: newest architectures providing super-linear scaling.
  - More robust and accurate moving parts algorithms in development.
    - Lagrangian Frame for grid movement.
  - Conjugate Heat Transfer
    - More accurate prediction in wall film and its effects on combustion and emissions under PCCI conditions with strong wall impingement.
  - Validation in progress for all flow regimes
    - With Multi-Species
    - Starting spray and chemistry model incorporation.
- **Cut-Cell grid Generation and Implementation**
  - Quickly generate grids from CAD surfaces of complex domains. New algorithm has been developed, more generic
  - Cubit Grid interface being developed for boundary conditions implementation.
  - Discussions with Sandia about incorporating LANL cut-cell ideas into Cubit

Technical Back-Up Slides

(Note: please include this “separator” slide if you are including back-up technical slides (maximum of five). These back-up technical slides will be available for your presentation and will be included in the DVD and Web PDF files released to the public.)

Fractional Step or Predictor Corrector Petrov-Galerkin and/or Characteristic Terms

- FEM Discretization for PCS or CBS
  - Velocity predictor

$$\left\{\Delta \mathbf{U}_i^{*}\right\} = -\Delta t\left[\mathbf{M}_v^{-1}\right]\left[\left[\mathbf{A}_u\right]\left\{\mathbf{U}_i\right\} + \left[\mathbf{K}_{\tau u}\right]\left\{\mathbf{U}_i\right\} - \left\{\mathbf{F}_{v_i}\right\} - \frac{\Delta t}{2}\left(\left[\mathbf{K}_{char}\right]\left\{\mathbf{U}_i\right\} - \left\{\mathbf{F}_{char_i}\right\}\right)\right]^n$$

where $\left\{\Delta U_i^{*}\right\} = \left\{U_i^{*}\right\} - \left\{U_i^{n}\right\}$

  - Velocity corrector (*desire this*)

$$U^{n+1} - U^{*} = \Delta t \frac{\partial P'}{\partial x_i}$$ and $\left\{U_i^{*}\right\}$ is an intermediate

  - How do we arrive at a corrector preserving mass/continuity?
    - Continuity

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_i}{\partial x_i} = -\frac{\partial U_i}{\partial x_i} \qquad \frac{\rho^{n+1} - \rho^{n}}{\Delta t} = -\frac{\partial U'_i}{\partial x_i}$$

Define $U' = \theta_1 U^{n+1} + \left(1-\theta_1\right)U^{n}$ with a level of implicitness

Desire $U^{n+1} - U^{*} = \Delta t \dfrac{\partial P'}{\partial x_i}$ Let $U'_i = \theta_1\left(-\Delta t \dfrac{\partial P'}{\partial x_i} + U_i^{*}\right) + \left(1-\theta_1\right)U_i^{n}$

Then $$\frac{1}{c^2}\Delta P = \Delta \rho = -\Delta t \frac{\partial U'_i}{\partial x_i} = -\Delta t \frac{\partial}{\partial x_i}\left[\left(\theta_1\left(-\Delta t\right)\frac{\partial P'}{\partial x_i} + \theta_1 U_i^{*}\right) + \left(1-\theta_1\right)U_i^{n}\right]$$

Density Solve (Pressure when incompressible flow)

So $$\frac{1}{c^2}\Delta P = \Delta\rho = -\Delta t\frac{\partial U_i^{'}}{\partial x_i} = \left[\left(\Delta t^2\theta_1\frac{\partial^2 P^{'}}{\partial x_i^2} - \Delta t\theta_1\frac{\partial U_i^*}{\partial x_i}\right) - \Delta t\left(1-\theta_1\right)\frac{\partial U_i^n}{\partial x_i}\right]$$

Let $P^{'} = \theta_2 P^{n+1} + (1-\theta_2)P^n$ with some level of implicitness

recall $$\Delta U^* = U^* - U^n$$

Then $$\frac{1}{c^2}\Delta P = \Delta\rho = -\Delta t\frac{\partial U_i^{'}}{\partial x_i} = \Delta t^2\theta_1\left(\theta_2\frac{\partial^2 P^{n+1}}{\partial x_i^2} + \left(1-\theta_2\right)\frac{\partial^2 P^n}{\partial x_i^2}\right) - \Delta t\left(\theta_1\frac{\partial \Delta U_i^*}{\partial x_i} + \frac{\partial U_i^n}{\partial x_i}\right)$$

and $\Delta P = P^{n+1} - P^n$

Density then $$\Delta\rho - \theta_2\frac{\partial^2\Delta P}{\partial x_i^2} = \frac{1}{c^2}\Delta P - \theta_1\theta_2\frac{\partial^2\Delta P}{\partial x_i^2} = \Delta t^2\theta_1\frac{\partial^2 P^n}{\partial x_i^2} - \Delta t\left(\theta_1\frac{\partial \Delta U_i^*}{\partial x_i} + \frac{\partial U_i^n}{\partial x_i}\right)$$

FEM Matrix form $$\left(\left[\mathbf{M}_p\right] + \Delta t^2 c^2\theta_1\theta_2\mathbf{H}\right)\left\{\Delta\rho_i\right\} = \left(\left[\frac{\mathbf{M}_p}{c^2}\right] + \Delta t^2\theta_1\theta_2\mathbf{H}\right)\left\{\Delta P_i\right\} =$$
$$\Delta t^2\theta_1\mathbf{H}\left\{P_i^n\right\} - \Delta t\left(\theta_1\mathbf{G}\left\{\Delta\mathbf{U}_i^*\right\} + \mathbf{G}\left\{\mathbf{U}_i^n\right\}\right) - \Delta t\left\{\mathbf{F}_{P_i}\right\}$$

Momentum/Velocity Corrector

Now $$P^{n+1} = \Delta P + P^{n}$$

recall $$P^{'} = \theta_2 P^{n+1} + (1-\theta_2)P^{n} = \theta_2 \Delta P + P^{n}$$

Then $$\Delta U_i = U^{n+1} - U^{n} = \Delta U^{*} - \Delta t \frac{\partial P^{'}}{\partial x_i} = \Delta U^{*} - \Delta t \left(\theta_2 \frac{\partial \Delta P}{\partial x_i} + \frac{\partial P^{n}}{\partial x_i} \right)$$

FEM Matrix form $$\left\{ \Delta \mathbf{U}_i \right\} = \left\{ \Delta \mathbf{U}^{*} \right\} - \Delta t \left[\mathbf{M}_u{}^{-1} \right] \left(\theta_2 \left[\mathbf{G} \right] \left\{ \Delta p_i \right\} + \left[\mathbf{G} \right] \left\{ p_i{}^{n} \right\} \right)$$

where $$\left\{ \mathbf{U}_i{}^{n+1} \right\} = \left\{ \Delta \mathbf{U}_i \right\} + \left\{ \mathbf{U}_i{}^{n} \right\}$$

final mass conserving velocity $$u^{n+1} = U^{n+1} / \rho^{n+1}$$

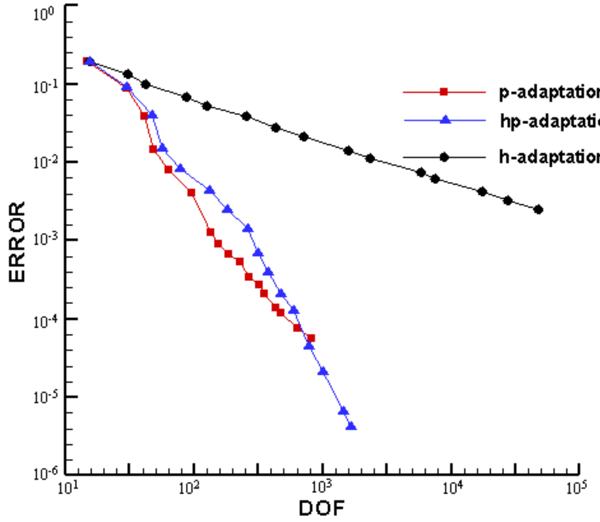
hp-adaptive methods for KIVA a CBS FEM method

- Why *hp*-adaptive grid
  - The use of *h*-adaptation can yield accurate solutions and rapid convergence rates.
    - Important when encountering singularities in the problem geometry.
  - Exponential convergence when higher-order, hp-adaptation
  - Error bounded by the following well known relation

$$\left\| u - u_h \right\|_m \leq c h^{k+1-m} \left\| u \right\|_r$$

'*u*' is assumed smooth in an H^{k+1} Sobolev norm, *m* is norm space, *r=k+1*, degree of integrable derivates in *H.*

- Convergence of *hp* about same as *p*. Speed of solution is better for *hp,* since the higher-order polynomials are used judiciously.
  - First perform h, then p for an hp scheme



Adaptation and Error – the driver for resolution

2011 DOE Merit Review

$$\|e_V\| = \left(\int_{\Omega} e_V^T e_V \, d\Omega \right)^{1/2}$$

L_2 norm of error measure

$$\|e_V\|^2 = \sum_{i=1}^{m} \|e_V\|_i^2$$

Element error

$$\eta_V = \left(\frac{\|e_V\|^2}{\|V^*\|^2 + \|e_V\|^2} \right)^{1/2} \times 100\%$$

Error distribution

$$\bar{e}_{avg} = \bar{\eta}_{\max} \left[\frac{\left(\|V^*\|^2 + \|e_V\|^2 \right)}{m} \right]^{1/2}$$

Error average

$$\xi_i = \frac{\|e\|_i}{\bar{e}_{avg}}$$

Refinement criteria

$$p_{new} = p_{old}\, \xi_i^{1/p}$$

Level of polynomial for element

- Error measures:
  - Residual, Stress Error, etc..
- Typical error measures:
  - Zienkiewicz and Zhu Stress
  - Simple Residual
  - Residual measure
    - How far the solution is from true solution.
    - "True" measure in the model being used to form the residual.
    - If model is correct, e.g., Navier-Stokes, then this is a measure how far solution is from the actual physics!